The Oseen resistance of a particle of arbitrary shape

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Let D_0 be the Stokes drag on an arbitrary body moving parallel to a principal axis of resistance, with velocity U, through an unbounded fluid. The Oseen drag, D, experienced by this same body moving with equal velocity and identical orientation through the unbounded fluid is then given by the expression

$$\frac{D}{D_0} = 1 + \frac{D_0}{16\pi\mu c U} R + O(R^2), \tag{1}$$

where c is any characteristic particle dimension and $R = cU\rho/\mu$ is the particle Reynolds number. An analogous expression is given for the case where the motion is not parallel to a principal axis. Finally, an expression is given for the Oseen resistance of an arbitrary particle falling parallel to a principal axis of resistance along the axis of a cylindrical tube of finite radius.

1. Introduction

The parameter of central importance in the hydrodynamics of low Reynolds number flows is the resistance experienced by a particle moving uniformly through an infinite fluid. Analytical resistance formulae are available in the Stokes régime for a diverse variety of particle shapes (Stimson & Jeffery 1926; Oseen 1927; Ghosh 1927; Relton 1931; Pell & Payne 1959; Payne & Pell 1960). Where analytical results are lacking, experimental data often exist for this régime (cf. Pettyjohn & Christiansen 1948; Heiss & Coull 1952; Becker 1959). The corresponding state of affairs in the Oseen régime is not nearly as satisfactory, and resistance formulae are available for only a relatively few three-dimensional particles (Oseen 1927).

In this paper we shall demonstrate that the Oseen resistance of an arbitrary body can be determined, at once, whenever the corresponding Stokes resistance is known for the particle. The method yields results correct only to terms of $O(R)$. This is not a real limitation, for Proudman & Pearson (1957) have shown that the Oseen equations correctly represent the asymptotic behaviour of the complete Navier–Stokes equations only to terms of this order of magnitude anyway; thus, higher-order solutions of the Oseen equations, such as that given by Goldstein (1929) for the sphere, are devoid of real physical significance.

2. Motion parallel to a principal axis

A closely related problem has been considered by Chang (1960) for the axially symmetric Stokes flow of a conducting fluid past a body of revolution in the presence of a uniform magnetic field. An equation identical to that cited in the abstract results, except that the (dimensionless) Hartmann number, M, appears in place of the Reynolds number, R. Chang (1960) gives a formal proof of this relation, as well as similar relations (1961 *a*, *b*), based on 'matching' the fundamental solution of the Stokes equations to the fundamental solution of his magnetohydrodynamic differential equation. These fundamental solutions correspond physically to the fields resulting from a *point* force concentrated at the origin, and are termed 'inner' and 'outer' solutions, respectively. The basic ideas derive from the work of Lagerstrom & Cole (1955) and Proudman & Pearson (1957).

Now, the structure of Oseen's equation and its fundamental solution are extremely similar to those for the corresponding magnetohydrodynamic problem.† Moreover, Proudman & Pearson (1957) have fully discussed the 'matching' of the Oseen and Stokes equations for the particular case of a spherical particle. For these reasons, it is unnecessary to give the formal details of our procedure; rather, the combined work of Proudman & Pearson (1957) and Chang (1960) assure us of the existence of a general relation of the form of equation (1). The only detail requiring further attention is the justification of the numerical coefficient of 16π in equation (1). On account of the generality of the result, the coefficient can be deduced from any one of the known solutions of the Oseen equation, e.g. a sphere. For a sphere of radius a, we have (Oseen 1927)

$$D = 6\pi\mu aU\left(1+\frac{3}{8}\frac{aU\rho}{\mu}\right)+O\left(\frac{aU\rho}{\mu}\right)^2.$$

Since the Stokes drag for the sphere is related to its radius by the expression $D_0 = 6\pi\mu aU$ we find, upon eliminating a between these two relations, that

$$\frac{D}{D_0} = 1+\frac{D_0}{16\pi\mu U}\left(\frac{\rho U}{\mu}\right)+O(R^2). \tag{2}$$

This can be put in the form of equation (1) by multiplying and dividing the inertial coefficient by any characteristic particle dimension, c.

There is one important difference with regard to the limits of applicability of Chang's (1960) result and ours. His result is restricted to axially symmetric flows because of the requirement that there be sufficient symmetry to preclude the existence of an electric field. On the other hand, our result is not limited to axisymmetric flows. Rather, it is limited only by the requirement that the Stokes force on the particle (and thus the Oseen force) be parallel to its direction of motion. Alternatively if the particle is at rest, the force must be parallel to the free-stream velocity. This will occur, for example, when the particle possesses three mutually perpendicular symmetry planes, providing that the motion is

† They do, however, differ in one important respect. The first-order Stokes (inner) equation (Proudman & Pearson 1957, equation (3·40)), which eventually gives rise to the drag term of $O(R)$, contains an inhomogeneous forcing term. This is lacking in the analogous first-order inner equation of Chang (1960), equation (6 *b*), which is the source of the drag term of $O(M)$ in the magnetohydrodynamic problem. However, as pointed out by the former authors, this inhomogeneous term does not give rise to a net drag. Hence, the analogy persists in spite of the difference.

normal to any one of them. Ellipsoids and right-angled prisms are examples of such bodies.

The criterion that the force and velocity be parallel will also be met by *arbitrary* bodies in certain circumstances; that is, by bodies devoid of any symmetry whatsoever. For, as pointed out by Landau & Lifshitz (1959), the Stokes vector force, $\mathbf{D}_0$, on an arbitrary body past which fluid streams with velocity $\mathbf{U}$ can be expressed by a linear relation of the form

$$\mathbf{D}_0 = \mathbf{\Phi} . \mathbf{U}. \tag{3}$$

Here, $\mathbf{\Phi}$ is a *symmetric* tensor, directly proportional to the viscosity of the fluid, but otherwise dependent only on the geometry of the particle. It follows at once from the properties of symmetric tensors that every arbitrary particle possesses three mutually perpendicular axes such that, if the motion is parallel to any one of them, the force, $\mathbf{D}_0$, will be parallel to $\mathbf{U}$. (We shall refer to these as the principal axes of resistance.) In consequence, equation (1) applies even to arbitrary bodies providing that the motion is parallel to a principal axis.

By way of illustration, for a circular disk of radius c, broadside to the stream, the Stokes drag is $D_0 = 16\mu c U$ (Lamb 1932). Thus from equation (1)

$$D = 16\mu c U(1 + R/\pi) + O(R^2),$$

in accord with Oseen's (1927) result. In a similar vein, when the disk is edge-on to the stream, $D_0 = \frac{32}{3}\mu c U$ (Lamb 1932), and we find for the Oseen drag

$$D = \tfrac{32}{3}\mu c U(1 + \tfrac{2}{3}R/\pi) + O(R^2),$$

again in agreement with Oseen's calculations. It is worth while noting that the latter orientation does not give rise to an axisymmetric motion. Equation (1) shows similar agreement with Oseen's calculations for the general ellipsoidal particle, providing that the motion is perpendicular to a symmetry plane.

The Oseen drag on a spherical particle appears to conform with experiment only up to $R \approx 1$ (Becker 1959). This can hardly be regarded as a major practical improvement over Stokes's original result. However, Carrier (1953) has shown empirically that predictions made on the basis of Oseen's equation come into excellent agreement with experimental data for spheres, cylinders and flat plates, up to $R \approx 20$, if the velocity $\mathbf{U}$ in Oseen's equation is multiplied by 0·43. The general applicability of Carrier's hypothesis would greatly enhance the value of the present work. And it therefore seems a worthwhile task to review critically the great mass of low Reynolds number settling velocity data with this thought in mind.

3. Motion oblique to a principal axis

Equation (1) is inapplicable when the orientation of the body is such that the stream velocity is not parallel to a principal axis. For now, the particle will also experience lateral forces, that is, normal to the direction of flow. However, the analogue of equation (1) can be developed for this case by availing ourselves of the detailed solution of the Oseen equations (Oseen 1927), for the motion of a general ellipsoid whose symmetry planes are arbitrarily inclined with respect to its direction of motion. This is done by eliminating the dimensions of the three semi-axes of the ellipsoid through their connexion with the Stokes drag.

If

$$\boldsymbol{\phi} = \boldsymbol{\Phi}/(\mu c) \tag{4}$$

denotes the (dimensionless) Stokes resistance tensor, appropriate to an arbitrary particle, the final expression for the Oseen force on the particle takes the form (in dyadic notation)

$$\mathbf{D} = \mu c U[\mathbf{I} + (R/32\pi)\{3\boldsymbol{\phi} - \mathbf{I}(\mathbf{U}/U).\boldsymbol{\phi}.(\mathbf{U}/U)\}].\boldsymbol{\phi}.(\mathbf{U}/U) + O(R^2), \tag{5}$$

where $\mathbf{I}$ is the idemfactor, $R = cU\rho/\mu$ is the particle Reynolds number and c is any characteristic length. Formal justification of this result can be provided along lines similar to those developed by Chang (1960) for the corresponding magnetohydrodynamic problem. The fundamental solutions are, naturally, more involved. The 'inner', Stokes solution is already available in Lamb's (1932) treatise.

Equation (5) can be expressed in terms of the three scalar components of the Oseen force. Let (x', y', z') be a system of Cartesian co-ordinates measured along the principal axes of resistance of the particle. If $(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ are unit vectors in these directions, then

$$\mathbf{I} = \mathbf{i}'\mathbf{i}' + \mathbf{j}'\mathbf{j}' + \mathbf{k}'\mathbf{k}'$$

and

$$\boldsymbol{\phi} = \mathbf{i}'\mathbf{i}'L_0 + \mathbf{j}'\mathbf{j}'M_0 + \mathbf{k}'\mathbf{k}'N_0, \tag{6}$$

where the essentially positive scalars (L_0, M_0, N_0), are the principal Stokes resistances, expressed in dimensionless form.

Now, let (x, y, z) be a second system of Cartesian co-ordinates having the same origin as the former, associated with the direction of the streaming flow, and let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be the corresponding unit vectors. Without loss in generality, suppose that the fluid streams in the positive z-direction with velocity U; that is $\mathbf{U} = \mathbf{k}U$, where U is essentially positive. If

$$\mathbf{D} = \mathbf{i}D_x + \mathbf{j}D_y + \mathbf{k}D_z \tag{7}$$

is the Oseen force exerted by the fluid *on* the particle, and if

$$a_{12} = \mathbf{i}.\mathbf{j}' = \cos(\mathbf{i}, \mathbf{j}'), \quad a_{21} = \mathbf{j}.\mathbf{i}' = \cos(\mathbf{j}, \mathbf{i}'), \quad \text{etc.}$$

denote the direction cosines, then a relatively simple calculation gives, for the components of force in the (x, y, z) system,

$$\left.\begin{aligned} D_x &= \mu c U(a_{11}\tilde{L} + a_{12}\tilde{M} + a_{13}\tilde{N}) + O(R^2), \\ D_y &= \mu c U(a_{21}\tilde{L} + a_{22}\tilde{M} + a_{23}\tilde{N}) + O(R^2), \\ D_z &= \mu c U(a_{31}\tilde{L} + a_{32}\tilde{M} + a_{33}\tilde{N}) + O(R^2), \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} \tilde{L} &= a_{31}L_0[1 + (3R/32\pi)(L_0 - Q_0)], \\ \tilde{M} &= a_{32}M_0[1 + (3R/32\pi)(M_0 - Q_0)], \\ \tilde{N} &= a_{33}N_0[1 + (3R/32\pi)(N_0 - Q_0)], \end{aligned}\right\} \tag{9}$$

in which

$$Q_0 = \tfrac{1}{3}(a_{31}^2 L_0 + a_{32}^2 M_0 + a_{33}^2 N_0). \tag{10}$$

It is worth while noting that the force experienced by the particle can also be expressed in (x', y', z')-co-ordinates in the form

$$\mathbf{D} = \mu c U(\mathbf{i}'\tilde{L} + \mathbf{j}'\tilde{M} + \mathbf{k}'\tilde{N}). \tag{11}$$

As a specific illustration, consider streaming flow past a circular disk of radius c, as in figure 1. The normal to the plane of the disk corresponds to the z'-axis, and the angle between this and the direction of motion of the stream is denoted by θ, where it is assumed that $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The y-axis is chosen so as to lie in the plane of the disk and to coincide with the y'-axis. By these means, the x'- and z'-axes may be regarded as deriving from the x- and z-axes, respectively, by a rotation about the y, y'-axis through the angle θ. The x- and z-axes then lie in the same plane as the corresponding primed axes. To be unambiguous, we also specify that the angle between the x'- and z-axes lie between 0 and $\frac{1}{2}\pi$. In figure 1, the y, y'-axis is directed out of the plane of the paper, at the reader, thereby leading to right-handed co-ordinate systems.

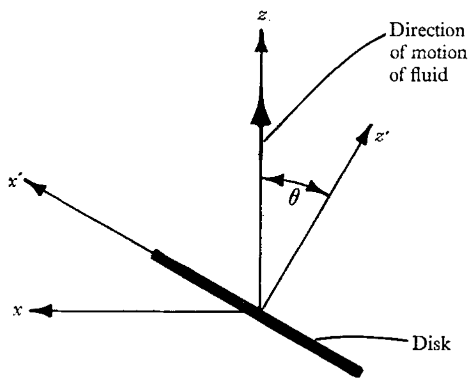

FIGURE 1. Streaming flow past a circular disk.

The direction cosines appropriate to the rotation are given by

$$a_{11} = a_{33} = \cos\theta, \quad a_{31} = -a_{13} = \sin\theta, \quad a_{22} = 1.$$

All others are zero. The principal Stokes resistances can be obtained from the results cited earlier for disks oriented broadside and edgewise to the stream. They are $L_0 = M_0 = \frac{32}{3}$, $N_0 = 16$. These lead to the following expressions for the components of the Oseen force on the disk:

$$D_x = -\tfrac{16}{3}\mu c U \sin\theta\cos\theta[1 + (R/\pi)(2 + \tfrac{1}{6}\sin^2\theta)] + O(R^2), \quad D_y = 0,$$

$$D_z = 16\mu c U[1 - \tfrac{1}{3}\sin^2\theta + (R/\pi)(1 - \tfrac{1}{2}\sin^2\theta - \tfrac{1}{18}\sin^4\theta)] + O(R^2).$$

The algebraic signs indicate that the forces act in the $-x$ and $+z$ directions respectively.

4. Settling of an arbitrary particle along the axis of a circular cylinder

A recent paper by Chang (1961*b*) gives the solution of the problem of the axially symmetric Stokes fall of an arbitrary body of revolution along the axis of a circular cylindrical tube filled with conducting fluid, in the presence of a magnetic field. Chang's results simultaneously take account of the first-order corrections to Stokes law necessitated by the presence of the tube walls and of the magnetic field. The increased resistance can be neatly calculated solely from a knowledge of the ordinary Stokes drag, D_0—that is, in the absence of both the tube walls and magnetic field—providing that c/l and M are both small. Here, l is the tube radius and M is the Hartmann number.

The analogous problem, in which the particle Reynolds number, R, replaces M, corresponds to the settling of a particle at the axis of a tube in the Oseen régime (in the absence of the magnetic field). This problem was solved long ago by Faxen (Oseen 1927, p. 198) for the case of a *spherical* particle at the axis. On the basis of our earlier remarks regarding the fundamental similarity of the Oseen and inertialess magnetohydrodynamic equations, we can at once generalize Faxen's formula and make it apply to an *arbitrary* particle falling parallel to a principal axis along the longitudinal cylinder axis. This only requires that we replace the sphere radius a in Faxen's formula by $(D_0/6\pi\mu U)$, with the result that the Oseen drag is

$$D = D_0 \Big/ \left[1 - \frac{D_0}{16\pi\mu c U} R - \frac{D_0}{6\pi\mu c U}\left(\frac{c}{l}\right) L(\tfrac{1}{2}Rl/c) + O\left(\frac{c}{l}\right)^3\right] + O(R^2), \qquad (12)$$

where $R = cU\rho/\mu$ is the particle Reynolds number. In addition, $L(x)$ is a function of argument $x = Rl/2c$, some values of which are tabulated in Oseen's book, i.e. for small x,

$$L(x) = 2{\cdot}104 - \tfrac{3}{4}x + \ldots,$$

and $L(0) = 2{\cdot}104$, $L(0{\cdot}5) = 1{\cdot}76$, $L(1) = 1{\cdot}48$, $L(2) = 1{\cdot}04$, $L(5) = 0{\cdot}46$.

Chang's (1961*b*) result is virtually identical to equation (12) in which the Hartmann number, M, replaces the particle Reynolds number, R. The primary difference between the two lies in the fact that in Chang's (1961*b*) formula, the denominator of equation (12) appears in the numerator, there being corresponding changes in the algebraic signs. Faxen's result is the more accurate of the two, at least in the special case where $R \to 0$, in which event equation (12) takes the form

$$D = D_0 \Big/ \left[1 - 2{\cdot}104 \frac{D_0}{6\pi\mu U l} + O\left(\frac{c}{l}\right)^3\right],$$

a result previously given by Brenner (1961*a*) for the Stokes régime.

It appears further that Faxen's integral, $L(x)$, is different from a corresponding integral, $K(x)$, appearing in Chang's treatment. They are, however, closely related in form, and both give the limiting values

$$L(0) = (3/\pi)\,K(0) = 2{\cdot}104 \quad \text{and} \quad L(\infty) = (3/\pi)\,K(\infty) = 0.$$

The last value shows that equation (12) becomes identical to equation (1), correctly to terms of $O(R)$, in the case where $c/l \to 0$.

Although equation (12) appears analytically sound, its practical applicability in the case of spherical particles has been questioned (Fayon & Happel 1960).

One last point worth noting concerns the pressure difference, $\Delta P > 0$, at a great distance on either side of the particle settling in the tube. Chang's (1961*b*) calculations show that

$$\Delta\mathbf{P}\pi l^2 = 2\mathbf{D},$$

where $\Delta\mathbf{P}$ is a vector pointing in the direction of diminishing pressure and having the magnitude of the pressure difference. This result confirms Brenner's (1961*b*) prediction of the existence of such a relation in the Oseen régime, it already having been demonstrated to hold in the Stokes régime (Brenner 1961*b*).

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